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Axisymmetric Solid with Non-Axisymmetric Load Using Matlab

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Abstract—In general rocket structural configurations are symmetric with respect to axis. Most of the situations the loading that comes on the structures are also symmetric. This makes the problem simple since axi-symmetric solid with axi-symmetric loading can be used for the structural analysis. But there are special situation where loading alone is not symmetric, it may be mechanical or thermal. To solve such a problem either three dimensional solid element can be used with loading at the appropriate locations. This will provide the complete solution at the expense of extra time on modeling and computation cost. This can be solved by another way where modeling of structural configuration becomes 2D but the non-axisymmetric load has to be expressed in the Fourier series form. Using this Fourier series, solution has to be obtained for each Fourier component. This solution can be obtained using two dimensional configurations. The total solution can be obtained by adding the solution obtained for each Fourier coefficient. Using the total solution, solution at desired angle can be obtained. The solution can be in the form of displacements, strains and stresses and reaction at the support points. This procedure considerably reduces the computation cost and modeling effort. It is proposed to develop finite element code using MATLAB for mechanical load. The results obtained with present code is validated with commercially available finite element software.

Index Terms—Axisymmetric solid, Finite element software,Fourier coefficient, Fourier series, MATLAB, Mechanical load, Non-axisymmetric load

1 Introduction

INthe aerospace industry, the stress analysis of complex axi-symmetric structures subjected to thermal and mechanical loads is of considerable interest. Generally the material properties and loadings are symmetric with respect to the axis. In such situations axisymmetric solid with axisymmetric loading can be employed. If the loading is axisymmetric, the displacement vector has two components in the radial and axial directions. This simplification is not possible when the structural configuration and material properties are symmetric with respect to the axis except for the loading. This can be addressed by modelling the complete structural configuration using 3-Dimensional model and applying the load at the appropriate location in the configuration. This will provide the solution in the form of displacements, strains and stresses at one shot at the cost of extra modelling and computation time. The other simplification in the modelling is reduced as two dimensional model having radial, axial and tangential degrees of freedom. The Non-symmetric load which is applied at the portion of the structural configuration has to be expressed in Fourier series form which represents the loading pattern. Then for each Fourier component the two-dimensional solution is obtained and summing up to the solution with appropriate symmetric and anti-symmetric component multiplication. Using this general solution, solution at desired angle between 0 to 360 can be obtained in the form of displacement, strain and stress. To validate the code a cylinder subjected to load at some portion of the configuration is applied and the results obtained with the present and NISA general purpose code were compared and the results obtained are provided in the tabular form. A practical problem in rocket industry is solved to establish robustness of the FE code developed. The structural configuration chosen for the analysis in nozzle end segment of solid rocket boosters subjected to four different load cases and the results obtained are provided in a graphical form. The results obtained are in good agreement for both the cases. For the convenient of the reader detailed finite element formulation is provided which will help those who have interest in knowing the formulation. The analysis of axisymmetric solid structures with non-symmetric load has to be expressed by representation of structural configuration using two-dimensional model and the non-symmetric load has to be expressed in the Fourier series form.

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2 Finite Element Formulation

Fourier series expression used to represent the load is given as

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (1)$$

Where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \quad n = 0, 1, 2, \ldots . \infty \quad (2)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \quad n = 1, 2, \ldots \infty \quad (3)$$

whereθ is the circumferential coordinate implied in the model. Since the load is expressed in the form Fourier series then the displacement field with in the body also has to be expressed in the series form consisting of cosine and sine terms. The displacement field within the body expressed in series form as follows

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$$u = \sum_{n=1}^{\infty} \left(u_{a_n} \cos n\theta + u_{b_n} \sin n\theta \right)$$
(4)

$$v = \sum_{n=1}^{\infty} \left(v_{a_n} \cos n\theta + v_{b_n} \sin n\theta \right) \tag{5}$$

$$w = \sum_{n=1}^{\infty} \left(w_{a_n} \sin n\theta + w_{b_n} \cos n\theta \right) \tag{6}$$

where u, v, w are the axial, radial and circumferential displacements respectively.
The strain displacement relation in polar co-ordinate system in defined as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \\ \gamma_{r\theta} \\ \gamma_{z\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial z} \\ \frac{u}{r} + \frac{1}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r} \\ \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \end{Bmatrix} \tag{7}$$

Using the principal of minimum potential energy stiffness and load vector corresponding to symmetric and anti-symmetric part are derived, the respective matrices and load vector are provided below.

2.1 Stiffness Matrix for Symmetric Components

Stiffness matrix for symmetric component of Fourier loading is given for n=0 is

$$k_{a_0} = 2\pi \int_A \left[B_{a_0}^i\right]^T [D] \left[B_{a_0}^i\right] r dA \tag{8}$$

Where $[D]$is a 6 x 6 constitutive matrix relates stress and strain. Strain displacement relation matrix for symmetric component for n=0 is given as

$$\left[B_{a_0}^i\right] = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z} & 0 \\ \frac{N_i}{r} & 0 & 0 \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Stiffness matrix for symmetric component of Fourier loading is given for n=1, 2, 3 etc are

$$k_{a_n} = \pi \int_A \left[B_{a_n}^i\right]^T [D] \left[B_{a_n}^i\right] r dA \tag{10}$$

Where

$$\left[B_{a_n}^i\right] = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z} & 0 \\ \frac{N_i}{r} & 0 & +\frac{nN_i}{r} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & 0 \\ -\frac{nN_i}{r} & 0 & \left(\frac{\partial N_i}{\partial r} - \frac{N_i}{r}\right) \\ 0 & -\frac{nN_i}{r} & \frac{\partial N_i}{\partial z} \end{bmatrix} \tag{11}$$

2.2 Stiffness Matrix for Anti-Symmetric Components

Stiffness matrix for different of Fourier component for n=0, 1, 2, needs to computed.
Stiffness matrix for n=0 is

$$k_{b_0} = 2\pi \int_A \left[B_{b_0}^i\right]^T [D] \left[B_{b_0}^i\right] r dA \tag{12}$$

Where

$$\left[B_{b_0}^i\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial N_i}{\partial r} - \frac{N_i}{r}\right) \\ 0 & 0 & \frac{\partial N_i}{\partial z} \end{bmatrix} \tag{13}$$

Stiffness matrix for n=1, 2, 3 etc are

$$k_{b_n} = \pi \int_A \left[B_{b_n}^i\right]^T [D] \left[B_{b_n}^i\right] r dA \tag{14}$$

Where

$$\left[B_{b_n}^i\right] = \begin{bmatrix} \frac{\partial N_i}{\partial r} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z} & 0 \\ \frac{N_i}{r} & 0 & -\frac{nN_i}{r} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial r} & 0 \\ \frac{nN_i}{r} & 0 & \left(\frac{\partial N_i}{\partial r} - \frac{N_i}{r}\right) \\ 0 & \frac{nN_i}{r} & \frac{\partial N_i}{\partial z} \end{bmatrix} \tag{15}$$

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2.3 Load Vector for Symmetric Components

Load vector for symmetric Fourier component for n = 0 is written as

$$\left\{F_{a_0}\right\} = 2\pi \int_l [N]^T \begin{Bmatrix} F_{r_0} \\ F_{z_0} \\ 0 \end{Bmatrix} rdl \tag{16}$$

Load vector for other symmetric Fourier component such as n = 1, 2, 3 etc is given as

$$\left\{F_{a_n}\right\} = \pi \int_l [N]^T \begin{Bmatrix} F_{r_n} \\ F_{z_n} \\ 0 \end{Bmatrix} rdl \tag{17}$$

2.4 Load Vector for Anti-Symmetric Components

Load vector for anti-symmetric Fourier component for n = 0 is written as

$$\left\{F_{b_0}\right\} = 2\pi \int_l [N]^T \begin{Bmatrix} 0 \\ 0 \\ F_{\theta_0} \end{Bmatrix} rdl \tag{18}$$

Load vector for other anti-symmetric Fourier component such as n = 1, 2, 3 etc is given as

$$\left\{F_{a_n}\right\} = \pi \int_l [N]^T \begin{Bmatrix} F_{r_n} \\ F_{z_n} \\ F_{\theta_n} \end{Bmatrix} rdl \tag{19}$$

2.5 Strain and Stress Computations

Strain vector for three-dimension in polar co-ordinates is given as

$$\{\varepsilon\}^T = \{\varepsilon_r \quad \varepsilon_z \quad \varepsilon_\theta \quad \gamma_{rz} \quad \gamma_{r\theta} \quad \gamma_{z\theta}\} \tag{20}$$

$$\{\varepsilon\}^T = \sum_{n=0}^{N} \{[B_{an}][q_{an}]\} + \{[B_{bn}][q_{bn}]\} \tag{21}$$

Where $\{q_{an}\}$ and $\{q_{bn}\}$ are the elemental displacement corresponding to symmetric and antisymmetric loads. Stress vector is given as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{22}$$

Where

$$\left[B^i_{a_n}\right] = \begin{bmatrix} \frac{\partial N_i}{\partial r}\cos n\theta & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z}\cos n\theta & 0 \\ \frac{N_i}{r} & 0 & +\frac{nN_i}{r}\cos n\theta \\ \frac{\partial N_i}{\partial z}\cos n\theta & \frac{\partial N_i}{\partial r}\cos n\theta & 0 \\ -\frac{nN_i}{r}\sin n\theta & 0 & \left(\frac{\partial N_i}{\partial r} - \frac{N_i}{r}\right)\sin n\theta \\ 0 & -\frac{nN_i}{r}\sin n\theta & \frac{\partial N_i}{\partial z}\sin n\theta \end{bmatrix} \tag{23}$$

And

$$\left[B^i_{b_n}\right] = \begin{bmatrix} \frac{\partial N_i}{\partial r}\sin n\theta & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial z}\sin n\theta & 0 \\ \frac{N_i}{r} & 0 & -\frac{nN_i}{r}\sin n\theta \\ \frac{\partial N_i}{\partial z}\sin n\theta & \frac{\partial N_i}{\partial r}\sin n\theta & 0 \\ \frac{nN_i}{r}\cos n\theta & 0 & \left(\frac{\partial N_i}{\partial r} - \frac{N_i}{r}\right)\cos n\theta \\ 0 & -\frac{nN_i}{r}\cos n\theta & \frac{\partial N_i}{\partial z}\cos n\theta \end{bmatrix} \tag{24}$$

Using the above relation strain and stress at a particular 'θ' location can be obtained.

3 Numerical Study

Present Finite element code developed using MATLAB^R tool is validated with two problem one is a sample problem provided in NISA-verification manual and the other one is used in rocket industry. The results obtained are in very good agreement with each other.

Problem (1)

A hollow circular cylinder having 5 inches, 7 inches and 8 inches as inner, outer radius and the height respectively considered for the analysis. This cylinder is subjected to an external pressure having variation in the form P(θ) = (P_i/2) (1+cos(mθ)) per unit area (where m = 10/3 and $\frac{-\pi}{m} \le \theta \le \frac{\pi}{m}$) on a portion of the circumference. The total load over the whole cylinder is 10,000 lb/in. This loading pattern is expressed as Fourier series with the Fourier coefficients as:-

$$a_0 = 2\,\bar{P}_{tot} \qquad a_1 = 1.8800\,\bar{P}_{tot} \qquad a_2 = 1.5570\,\bar{P}_{tot}$$
$$a_3 = 1.1500\,\bar{P}_{tot} \qquad a_4 = 0.7090\,\bar{P}_{tot} \qquad a_5 = 0.3400\,\bar{P}_{tot}$$

Where $\bar{P}_{tot} = \frac{P_{tot}}{2\pi R}$, where R is the outer radius of the cylinder. Eight noded quadrilateral element is used for the discretization having 2 elements and 4 elements in the radial and longitudinal directions respectively. The analysis is repeated with six noded triangular elements and the results are comparable. Table 1 provides the displacement at different sectors at the top most outer point of the cylinder. Table 2 provides the stresses at different sectors at the cylinder inner most point. This problem is solved by fixing all the nodes at bottom points of the cylinder.

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TABLE 1
COMPARISON OF DISPLACEMENT AT THE TOP NODE OF THE OUTER SURFACE OF THE CYLINDER (10^{-3} INCH)

Angle (deg.)	Present			NISA		
	u	v	w	u	v	w
0	-3.02	0.82	0.00	-3.02	0.82	0.00
18	-2.49	0.69	0.40	-2.49	0.69	0.40
54	0.04	0.03	0.86	0.04	0.03	0.86
90	1.09	-0.26	0.63	1.09	-0.26	0.63
180	-0.17	0.05	0.00	-0.17	0.05	0.00

TABLE 2
COMPARISON AVERAGE NODAL STRESS (IN KIPS) AT DIFFERENT SECTORS AT BOTTOM NODE OF THE INNER SURFACE OF THE CYLINDER

angle		Sx	Sy	Sz	Sxy	Sxz	Syz
0	Present	-3.06	-8.72	-3.28	-2.61	0.00	0.00
	NISA	-3.06	-8.72	-3.28	-2.61	0.00	0.00
18	Present	-2.55	-7.26	-2.75	-2.06	-0.13	1.63
	NISA	-2.55	-7.26	-2.75	-2.06	-0.13	1.63
54	Present	-0.23	-0.72	-0.29	-0.03	-0.16	2.08
	NISA	-0.23	-0.72	-0.29	-0.03	-0.16	2.08
90	Present	0.62	1.72	0.65	0.42	-0.01	0.86
	NISA	0.62	1.72	0.65	0.42	-0.01	0.86
180	Present	-0.04	-0.09	-0.07	-0.05	0.00	0.00
	NISA	-0.04	-0.09	-0.07	-0.05	0.00	0.00

Problem (2)
A practical problem which is regularly used in rocket industry is studied to check the robustness of the code developed. Figure 1 shows the solid rocket booster subjected to different load cases as described below. Results of solid rocket boosters are obtained at different desired angle starting from 0 to 360 degrees. Four noded quadrilateral element with non-axisymmetric loading are used for the analysis.

LOAD CASES

Load case-1: Structure is analyzed for uniform internal pressure of 60 KSC acting on the inner surface of the structure. Fourier coefficient is used to represent the loading is. $a_0 =$ 1.0000

Load case-2: Structure is analyzed for internal pressure of 60 KSC with reduced end force acting on the inner surface of the structure.

Load case-3: Structure is analyzed for concentrated load of 10t acting at the node 526 on the outer surface of the structure at 135 degrees. Six Fourier coefficients are used to represents the given non symmetric loading.

Load case-4: Nozzle end segment of the booster is analyzed for the combined action of concentrated load (Nozzle actuation load) of 10t acting on the node 526 at the outer surface of the structure at 135 degrees and a pressure load of 60 KSC acting on the inner surface of the structure.

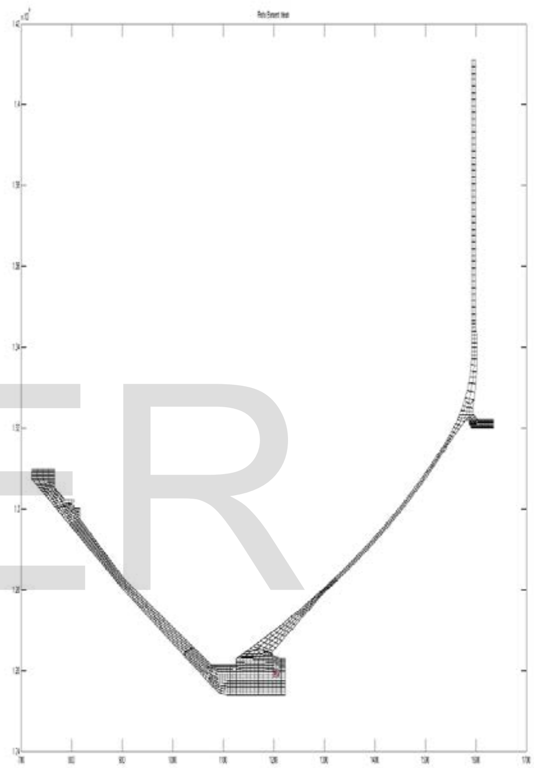

FIG. 1. FINITE ELEMENT MESH OBTAINED FROM MATLAB

RESULTS AND DISCUSSION

In the load case-1 and load case-2, the loading is also symmetric to the axis. Due to the symmetric loading, this is a case of axisymmetric solid with symmetric load. So the response of the structure due to the loading is same in all sectors. The graphs showing the results of displacements and stresses in the y-direction using both the softwares shows a straight line parallel to the x axis. Due to the symmetric loading the values of circumferential displacement and the stresses Sxz and Syz are zero in all sectors.

In the load case-3 and load case-4, the pressure loading is symmetric to the axis and the concentrated load is not symmetric to the axis. From the graphs of displacements and stresses, it can conclude that, the maximum value is obtained at 135 degree.

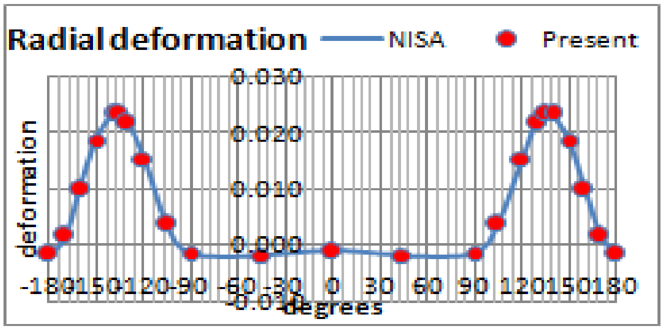

FIG. 2.COMPARISON OF DISPLACEMENT IN RADIAL DIRECTION (U) AT THE LOAD ACTING POINT IN LOAD CASE 3

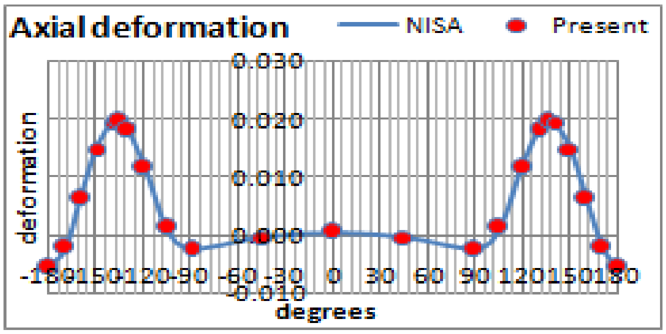

FIG. 3.COMPARISON OF DISPLACEMENT IN AXIAL DIRECTION (V) AT THE LOAD ACTING POINT IN LOAD CASE 3

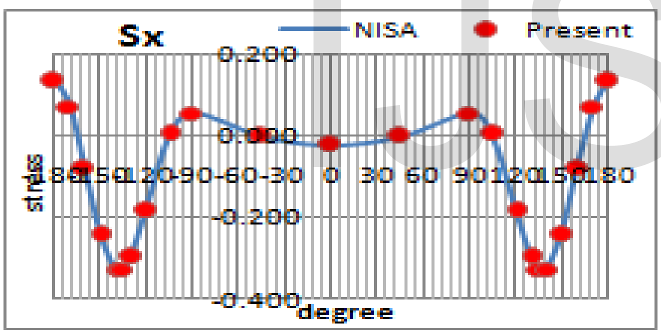

FIG. 4.COMPARISON OF STRESS SXAT THE LOAD ACTING POINT IN LOAD CASE 3

Fig. 2 and fig. 3 shows the comparison of displacement in radial and axial direction atthe load acting point in load case 3. Fig. 4 shows the comparison of stress Sx in load case 3. The practical study concludes that the response of the structure due to the concentrated and pressure loading is same in both studies. i.e. the results of displacement and stresses from the NISA software and present study shows the same values in each sectors. The present study was done by the MATLAB coding

4CONCLUSION

Finite element formulation for axi-symmetric solid with Fourier load developed and implemented in MATLAB. The developed code using the above formulation is validated with hollow circular cylinder with partial loading at the outer surface of the cylinder is considered. The non-symmetric load with respect to the axis is represented through Fourier series with six coefficients. Eight and Four noded quadrilateral element and six noded triangular elements are used for the modeling. Results obtained with present code are validated with NISA software at different sectors such as 0, 18, 54, 90 and 180. Both displacements and nodal average stresses matches well with each other. Using the present MATLAB code four practical load cases for a practical problem of rocket motor case are solved. Four noded quadrilateral elements are used for the modeling of the structure. Results from the present code are validated with NISA software at all sectors. The response of the structure due to the loading obtained from the both softwares is same at each sector. The displacements and stresses show the maximum value at the load acting sector.

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